

Minimization of an Isoperimetric Functional Perturbed by a Regular-Kernel Potential

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Received: 22 July 2026 Accepted: 23 September 2026 Published: 15 October 2026

DOI: 10.15849/ijasca.180 License: Open access under CC BY 4.0. Authors retain copyright.

Publisher: Al-Zaytoonah University of Jordan (ZUJ), Amman, Jordan.

Cite this article: I. Ouskhnid, "Minimization of an isoperimetric functional perturbed by a regular-kernel potential," *Int. J. Advance Soft Compu. Appl.*, vol. 18, no. 3, pp. 247–265, 2026, doi: <https://doi.org/10.15849/ijasca.180>.

ABSTRACT

In this paper, we investigate the minimization of the shape functional energy defined by the sum of the classical perimeter with a competing potential of a regular kernel, which is maximized by the ball. On the class of measurable subsets of the m -dimensional Euclidean space, which are of finite perimeter and have a fixed volume v , we prove the existence of a minimizer and we determine a condition on the kernel for which the ball becomes the minimizer of the considered problem. We then develop a MATLAB-based numerical algorithm to approximate the shape functional problem for the kernel exponential. The numerical results indicate the existence of a threshold area s_0 , which depends on the area and the regular kernel, such that below which the disk appears to minimize the shape functional among the considered shapes. Also, we augmented the considered shape functional by adding the Fraenkel asymmetry of E multiplied by $2v\|K\|_{L^1}$, we prove that the ball of volume v is a minimizer of the augmented shape functional.

Keywords: Classical perimeter; Riesz's Rearrangement inequality; Shape optimization; Monte Carlo method.

AMS Subject Classification (2020): 49Q10, 90C31.

1 Introduction

On the admissible class $\mathcal{A}_v$ constituted by measurable subsets $E \subset \Omega$ of finite perimeter (cf. Def. 1) whose Lebesgue measure $|E| = v$ endowed with the L^1 -topology, where Ω is a compact holdall admitting a non-empty interior $\overset{\circ}{\Omega}$ whose volume $|\overset{\circ}{\Omega}| > v > 0$ and with a Lipschitz boundary $\partial\Omega$, we consider the shape functional $\mathcal{G}$ defined as follows:

$$\forall E \in \mathcal{A}_v, \quad \mathcal{G}(E) = P(E) + \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_E(x) K(x-y) \chi_E(y) dx dy, \quad (1)$$

where $P(E)$ denotes the perimeter of the shape E , χ_E denotes its characteristic function and $K \in L^1(\mathbb{R}^m, dx)$ is a fixed non-negative kernel which will be assumed symmetric decreasing rearrangement invariant, i.e., $K^* = K$ (cf. [5, page 82]). As an example of kernels on $\mathbb{R}^m$ obeying these conditions, we can cite the exponential functions $K(x) := \exp(-\|x\|^\alpha)$ with $\alpha > 0$ and $\|x\|$ denotes the standard Euclidean norm in $\mathbb{R}^m$.

In the shape optimization literature, it is well-known that the ball of volume v , noted $B_{R_v} \subset \mathbb{R}^m$, minimizes the perimeter functional $E \mapsto P(E)$ on the class of shapes $\mathcal{A}_v$; on the other hand the same ball maximizes the second term (convolution) in the defining expression (1) of the functional $\mathcal{G}$ on $\mathcal{A}_v$ (cf. [5]). That is to say, these two terms constituting the shape functional $\mathcal{G}$ are competitive on the class $\mathcal{A}_v$.

This paper will be organized as follows: Section 2 is devoted to the study of the shape functional $\mathcal{G}$, in particular we show that $\mathcal{G}$ is lower semi-continuous on $\mathcal{A}_v$, and admits a minimizer on $\mathcal{A}_v$; Section 3 aims at computing the approximate value of the shape functional $\mathcal{P}_K(E) := \int_E \int_E K(x-y) dx dy$ by using an algorithm on MATLAB, for a tested regular shape in $\mathbb{R}^2$ of fixed area; in Section 4, we augment the shape functional $\mathcal{G}(E)$ by adding the term $2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E)$, where

$\alpha(E) := \inf_{y \in \mathbb{R}^m} \frac{|E \Delta (B_{R_v} + y)|}{|E|}$. We prove that the ball of volume v minimizes the augmented shape functional on $\mathcal{A}_v$.

2 Generalities on the Shape Functional $\mathcal{A}$

2.1 Sets of finite perimeter

In this paragraph, we recall definition and we give briefly some properties of the perimeter.

Definition 1. A nonempty measurable subset $E \subset \Omega$ is called *finite perimeter in De Giorgi's sense* (cf. [1], [4], [6]) when the following supremum is finite:

$$P(E) := \sup \left\{ \int_{\mathbb{R}^n} \chi_E(x) \operatorname{div}(F(x)) dx; F \in C_c^1(\mathbb{R}^m, \mathbb{R}^m), \|F\|_\infty \leq 1 \right\} < +\infty$$

In other words, a measurable subset $E \subset \Omega$ is considered finite perimeter if and only if the weak gradient of its characteristic function $\nabla \chi_E: C_c^1(\mathbb{R}^m, \mathbb{R}^m) \rightarrow \mathbb{R}$ is a bounded vectorial Radon measure with a finite total variation $\|\nabla \chi_E\| = P(E)$. Particularly, for a measurable subset E of finite Lebesgue volume, its perimeter in De Giorgi's sense will be finite if and only if the characteristic function χ_E belongs to the Banach space of functions of bounded variation $BV(\Omega)$ (cf. [4, page 194]).

The classical isoperimetric inequality states that the perimeter of any measurable set E with finite volume v is bigger than the perimeter of balls with the same volume v ; that is, for every admissible shape E belonging to the class $\mathcal{A}_v := \{E \subset \Omega: \chi_E \in BV(\Omega), |E| = |B_{R_v}| = v\}$ we have:

$$m \omega_m^{\frac{1}{m}} |E|^{\frac{m-1}{m}} \leq P(E) \quad (2)$$

where ω_m denotes the volume of unit ball B in the Euclidean space $\mathbb{R}^m$. In fact, since the perimeter of unit ball is $P(B) = m\omega_m$, then the left-hand side of (2) becomes $P(B_{R_v})$ where, henceforth,

$R_v := \left(\frac{v}{\omega_m}\right)^{\frac{1}{m}}$ will denote the radius of a ball $B_{R_v}(y)$ (resp. B_{R_v}) of volume v centered at $y \in \mathbb{R}^m$ (resp. at $0 \in \mathbb{R}^m$).

2.2 Basic properties of the regular potential $\mathcal{P}_k$

In this paragraph, the class of admissible shapes $\mathcal{A}_v$ will be endowed by the L^1 -topology. That is, a sequence of admissible shapes $E_k \in \mathcal{A}_v$ converges to an admissible shape $E \in \mathcal{A}_v$ if and only if the Lebesgue measure of the symmetric difference of the shapes E_k and E , $|E_k \Delta E| := \int_{\mathbb{R}^m} |\chi_{E_k}(x) - \chi_E(x)| dx$, converges to zero when k goes to infinity.

Let $K \in L^1(\mathbb{R}^m, dx)$ be a kernel, which will be by definition a non-negative function, symmetric decreasing rearrangement invariant, i.e., $K^* = K$ (cf. [5, page 82]). On the class of admissible shapes $\mathcal{A}_v$, we consider the functional defined by the following integral expression:

$$\forall E \in \mathcal{A}_v, \quad \mathcal{P}_K(E) := \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_E(x) K(x-y) \chi_E(y) dx dy$$

We start, then, our analysis by proving the continuity of the shape functional $\mathcal{P}_K$ on $\mathcal{A}_v$ with respect to the L^1 -topology.

Proposition 1. For any kernel $K \in L^1(\mathbb{R}^m)$, we have the following affirmations:

- (1) $\forall E, F \in \mathcal{A}_v, |\mathcal{P}_K(E) - \mathcal{P}_K(F)| \leq 2 \|K\|_{L^1} |E \Delta F|$.
- (2) $\mathcal{P}_K$ is continuous in the sense of characteristic functions (i.e. L^1 -topology).
- (3) The ball of volume v is a maximizer of the shape functional $\mathcal{P}_K$ on $\mathcal{A}_v$.

Proof. 1) For any pair of shapes E and $F \in \mathcal{A}_v$, by organizing the difference $\mathcal{P}_K(E) - \mathcal{P}_K(F)$, then we get

$$\begin{aligned} \mathcal{P}_K(E) - \mathcal{P}_K(F) &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\chi_E - \chi_F)(x) \chi_E(y) K(x-y) dy dx \\ &+ \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\chi_E - \chi_F)(y) \chi_F(x) K(x-y) dx dy. \end{aligned}$$

By Fubini's theorem, we also have

$$\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\chi_E - \chi_F)(x) \chi_E(y) K(x-y) dy dx = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^m} \chi_E(y) K(x-y) dy \right) (\chi_E - \chi_F)(x) dx$$

So, we get

$$\left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\chi_E - \chi_F)(x) \chi_E(y) K(x-y) dy dx \right| \leq \|K\|_{L^1} |E \triangle F|$$

In the same way, we obtain:

$$\left| \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} (\chi_E - \chi_F)(y) \chi_F(x) K(x-y) dx dy \right| \leq \|K\|_{L^1} |E \triangle F|$$

Therefore, we deduce that for all E and $F \in \mathcal{A}_v$ we have $|\mathcal{P}_K(E) - \mathcal{P}_K(F)| \leq 2 \|K\|_{L^1} |E \triangle F|$.

is a consequence of 1).

is none other than Riesz's rearrangement theorem (cf. [5, th. 3.7]). $\square$

Thanks to the lower semi-continuity of the perimeter (cf. [4, Th. 5.2]) and the continuity of $\mathcal{P}_K$, we can deduce that:

Corollary 1. *The shape functional (1), $\mathcal{G} := P + \mathcal{P}_K$, is lower semi-continuous on the class of shapes $\mathcal{A}_v$.*

2.3 Minimization of the shape functional $\mathcal{G}$ on $\mathcal{A}_v$

Recall that the kernel $K \in L^1(\mathbb{R}^m, dx)$ is assumed non-negative, then the following lower bound is a finite real number:

$$\inf\{\mathcal{G}(E) : E \in \mathcal{A}_v\} \geq 0$$

Proposition 2. *There exists an admissible set $E^* \in \mathcal{A}_v$ such that:*

$$\mathcal{G}(E^*) = \inf\{\mathcal{G}(E) : E \in \mathcal{A}_v\}$$

Proof. Let $m(v) := \inf\{\mathcal{G}(E) : E \in \mathcal{A}_v\}$. Then there exists a minimizing sequence $(E_k) \subset \mathcal{A}_v$ such that $\lim_{k \rightarrow +\infty} \mathcal{G}(E_k) = m(v)$. On the other hand, since the sequence of perimeters $P(E_k) \leq \mathcal{G}(E_k)$ is bounded, then this implies that the sequence of characteristic functions χ_{E_k} is bounded in the Banach space of functions of bounded variation $BV(\Omega)$ because we have

$$\|\chi_{E_k}\|_{BV} = |E_k| + P(E_k) \leq v + \mathcal{G}(E_k).$$

Thus, according to the compactness theorem for BV functions $BV(\Omega) \hookrightarrow L^1(\Omega)$ ([4, th. 5.5, p. 202]), there exists a measurable subset $E^* \subset \Omega$ of finite perimeter such that $\lim_{k \rightarrow +\infty} |E_k \triangle E^*| = 0$.

The volume constraint $|E^*| = v$ follows directly from the L^1 -convergence of the characteristic functions. Indeed, since $\chi_{E_k} \rightarrow \chi_{E^*}$ in $L^1(\mathbb{R}^m)$ we have

$$||E_k| - |E^*|| = \left| \int_{\mathbb{R}^m} (\chi_{E_k} - \chi_{E^*})(x) dx \right| \leq \int_{\mathbb{R}^m} |\chi_{E_k}(x) - \chi_{E^*}(x)| dx = \|\chi_{E_k} - \chi_{E^*}\|_{L^1(\mathbb{R}^m)}$$

Then, $|E^*| = \lim_{k \rightarrow \infty} |E_k| = v$, because $|E_k| = v$.

Since $E_k \subset \Omega$ for every k , then $\chi_{E_k} = 0$ a.e. on $\mathbb{R}^m \setminus \Omega$. Therefore,

$$\int_{\mathbb{R}^m \setminus \Omega} |\chi_{E_k}(x) - \chi_{E^*}(x)| dx = \int_{\mathbb{R}^m \setminus \Omega} |\chi_{E^*}(x)| dx = \int_{\mathbb{R}^m} \chi_{E^* \setminus \Omega}(x) dx = |E^* \setminus \Omega|$$

On the other hand, we have

$$|E^* \setminus \Omega| = \int_{\mathbb{R}^m \setminus \Omega} |\chi_{E_k}(x) - \chi_{E^*}(x)| dx \leq \int_{\mathbb{R}^m} |\chi_{E_k}(x) - \chi_{E^*}(x)| dx \leq \|\chi_{E_k} - \chi_{E^*}\|_{L^1(\mathbb{R}^m)},$$

the fact that χ_{E_k} converges to χ_{E^*} in L^1 , then $\|\chi_{E_k} - \chi_{E^*}\|_{L^1(\mathbb{R}^m)} \rightarrow 0$ as $k \rightarrow +\infty$. Hence, $|E^* \setminus \Omega| = 0$.

Consequently, $E^* \subset \Omega$. Therefore, the subset E^* belongs to $\mathcal{A}_v$. Finally, thanks to the lower semi-continuity of the functional $\mathcal{G}$ (cf. Cor. 1), we conclude that $\mathcal{G}(E^*) = \liminf_{k \rightarrow +\infty} \mathcal{G}(E_k) = m(v)$ and that the shape E^* is a minimizer of the shape functional $\mathcal{G}$ on $\mathcal{A}_v$. $\square$

In the rest of this paper, we will establish that: for any considered kernel $K \in L^1(\mathbb{R}^m, dx)$ the minimizer shape $E^* \in \mathcal{A}_v$ of the correspondent shape functional $\mathcal{G}$ is a ball of volume v under condition on the kernel K .

Theorem 1. Let $K \in L^1(\mathbb{R}^m)$ be a kernel. Then, there exists a constant universal $C_m > 0$ such that: if

$\|K\|_{L^1(\mathbb{R}^m)} \leq \frac{\sqrt{C_m P(B_{R_v})}}{2v} \inf_{E \in \mathcal{A}_v} (P(E) - P(B_{R_v}))^{\frac{1}{2}}$, then the ball of radius R_v minimizes the shape functional $\mathcal{G}$ on the admissible class $\mathcal{A}_v$.

Proof. For any $E \in \mathcal{A}_v$, we have $\mathcal{G}(B_{R_v}) - \mathcal{G}(E) = P(B_{R_v}) - P(E) + \mathcal{P}_K(B_{R_v}) - \mathcal{P}_K(E)$.

Remember that from the quantitative isoperimetric inequality (see [2, pr. 2.1] or [3, th. 4.3]), there exists a universal constant $C(m) > 0$ such that $P(E) \geq P(B_{R_v}) + C(m)P(B_{R_v})\alpha(E)^2$, where $\alpha(E)$ is the Fraenkel asymmetry of the shape E .

Thanks to Proposition 1 and the fact that the ball maximizes the shape functional $\mathcal{P}_K$, we obtain for every $y \in \mathbb{R}^m$

$$\mathcal{P}_K(B_{R_v}(y)) - \mathcal{P}_K(E) \leq 2 \|K\|_{L^1(\mathbb{R}^m)} |E \triangle B_{R_v}(y)|$$

Since the shape functional $\mathcal{P}_K$ is invariant up to translation, because for any measurable subset $E \subset \mathbb{R}^m$, we have

$$\begin{aligned} \forall a \in \mathbb{R}^m, \mathcal{P}_K(E + a) &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{E+a}(x) K(x-y) \chi_{E+a}(y) dx dy \\ &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_E(x-a) K((x-a)-(y-a)) \chi_E(y-a) dx dy = \mathcal{P}_K(E) \end{aligned}$$

This gives us: $\mathcal{P}_K(B_{R_v}) = \mathcal{P}_K(B_{R_v}(y))$.

Therefore,

$$\mathcal{P}_K(B_{R_v}) - \mathcal{P}_K(E) \leq 2 \|K\|_{L^1(\mathbb{R}^m)} \inf_{y \in \mathbb{R}^m} |E \triangle B_{R_v}(y)| \leq 2 \|K\|_{L^1(\mathbb{R}^m)} |E| \alpha(E)^2.$$

Hence,

$$\begin{aligned}
\mathcal{G}(B_{R_v}) - \mathcal{G}(E) &\leq P(B_{R_v}) - P(E) + 2 \|K\|_{L^1(\mathbb{R}^m)} |E| \left(\frac{P(E) - P(B_{R_v})}{C(m)P(B_{R_v})} \right) \\
&\leq (P(E) - P(B_{R_v}))^{\frac{1}{2}} \left(\frac{2 \|K\|_{L^1(\mathbb{R}^m)} |E|}{\sqrt{C(m)P(B_{R_v})}} - (P(E) - P(B_{R_v}))^{\frac{1}{2}} \right) \\
\forall E \in \mathcal{A}_v: \frac{2 \|K\|_{L^1(\mathbb{R}^m)} |E|}{\sqrt{C(m)P(B_{R_v})}} - (P(E) - P(B_{R_v}))^{\frac{1}{2}} &\leq 0 \Leftrightarrow \forall E \in \mathcal{A}_v: \frac{2 \|K\|_{L^1(\mathbb{R}^m)} |E|}{\sqrt{C(m)P(B_{R_v})}} \\
&\leq (P(E) - P(B_{R_v}))^{\frac{1}{2}} \\
&\Leftrightarrow \frac{2v \|K\|_{L^1(\mathbb{R}^m)}}{\sqrt{C(m)P(B_{R_v})}} \leq \inf_{E \in \mathcal{A}_v} (P(E) - P(B_{R_v}))^{\frac{1}{2}} \Leftrightarrow \|K\|_{L^1(\mathbb{R}^m)} \\
&\leq \frac{\sqrt{C(m)P(B_{R_v})}}{2v} \inf_{E \in \mathcal{A}_v} (P(E) - P(B_{R_v}))^{\frac{1}{2}}.
\end{aligned}$$

Thus, for every kernel K with $\|K\|_{L^1(\mathbb{R}^m)} \leq \frac{\sqrt{C(m)P(B_{R_v})}}{2v} \inf_{E \in \mathcal{A}_v} (P(E) - P(B_{R_v}))^{\frac{1}{2}}$, we obtain $\mathcal{G}(B_{R_v}) \leq \mathcal{G}(E)$ for any $E \in \mathcal{A}_v$.

Consequently, the ball B_{R_v} is a minimizer of the shape functional $\mathcal{G}$ for every kernel K satisfying

$$\|K\|_{L^1(\mathbb{R}^m)} \leq \frac{\sqrt{C(m)P(B_{R_v})}}{2} \inf_{E \in \mathcal{A}_v} (P(E) - P(B_{R_v}))^{\frac{1}{2}}. \quad \square$$

2.4 The first shape derivative of the shape functional $\mathcal{G}$

In this subsection, E denotes an open set of finite perimeter such that ∂E is a $\mathcal{C}^2$ hypersurface. Given a vector field $X \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^m)$, we define the flow induced by X as the solution in t of the ODEs

$$\begin{cases} \partial_t \phi_t(x) = X(\phi_t(x)), \\ \phi_0(x) = x \end{cases}$$

for every $x \in \mathbb{R}^m$. Let $E_t := \phi_t(E)$. We assume that the regular kernel K belongs to $\mathcal{C}^1(\mathbb{R}^m, \mathbb{R}^+)$.

Proposition 3. Let $X \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^m)$ be a smooth vector field such that $\operatorname{div} X = 0$, and define the perturbation $\phi_t(x) = x + tX(x)$, $E_t = \phi_t(E)$. We have

$$\frac{d}{dt} (\mathcal{P}_K(E_t))|_{t=0} = 2 \int_{\partial E} \int_E K(x-y) X(x) \cdot \nu(x) dy d\mathcal{H}^{m-1}(x)$$

Proof. We have $\mathcal{P}_K(E_t) = \int_{E_t} \int_{E_t} K(x-y) dx dy$. Applying a change of variables, we obtain,

$$\mathcal{P}_K(E_t) = \int_E \int_E K(\phi_t(x) - \phi_t(y)) \det(D\phi_t(x)) \det(D\phi_t(y)) dx dy.$$

We have $\det(D\phi_t) = 1 + t \operatorname{div}(X) + o(t^2)$. We differentiate with respect to t and set $t = 0$, we obtain

$$\begin{aligned}
\frac{d}{dt}(\mathcal{P}_K(E_t))|_{t=0} &= \int_E \int_E \nabla K(x-y) \cdot (X(x) - X(y)) \, dx \, dy + \int_E \int_E K(x-y) \operatorname{div}(X(x)) \, dx \, dy \\
&\quad + \int_E \int_E K(x-y) \operatorname{div}(X(y)) \, dx \, dy \\
&= \int_E \int_E \nabla K(x-y) \cdot X(x) \, dx \, dy - \int_E \int_E \nabla K(x-y) \cdot X(y) \, dx \, dy \\
&\quad + \int_E \int_E K(x-y) \operatorname{div}(X(x)) \, dx \, dy + \int_E \int_E K(x-y) \operatorname{div}(X(y)) \, dx \, dy
\end{aligned}$$

Since $\nabla K(x-y) = -\nabla K(y-x)$, then

$$\begin{aligned}
\frac{d}{dt}(\mathcal{P}_K(E_t))|_{t=0} &= \int_E \int_E \nabla K(x-y) \cdot X(x) \, dx \, dy + \int_E \int_E \nabla K(y-x) \cdot X(y) \, dx \, dy \\
&\quad + \int_E \int_E K(x-y) \operatorname{div}(X(x)) \, dx \, dy + \int_E \int_E K(x-y) \operatorname{div}(X(y)) \, dx \, dy
\end{aligned}$$

The fact that K is symmetric, it gives

$$\begin{aligned}
\frac{d}{dt}(\mathcal{P}_K(E_t))|_{t=0} &= 2 \int_E \int_E \left(\nabla K(x-y) \cdot X(x) + K(x-y) \operatorname{div}(X(x)) \right) \, dx \, dy \\
&= 2 \int_E \left(\int_E \operatorname{div}_x (K(x-y)X(x)) \, dy \right) \, dx
\end{aligned}$$

Applying the Gauss Divergence Theorem, we obtain

$$\frac{d}{dt}(\mathcal{P}_K(E_t))|_{t=0} = 2 \int_{\partial E} \int_E K(x-y) \, dy \, X(x) \cdot \nu(x) \, d\mathcal{H}^{m-1}(x)$$

Consequently, $d\mathcal{P}_K(E)[X] = 2 \int_{\partial E} \left(\int_E K(x-y) \, dy \right) X(x) \cdot \nu(x) \, d\mathcal{H}^{m-1}(x)$, where ν is the outward unit normal to ∂E . $\square$

Corollary 2. The first shape derivative under volume constraint of the shape functional $\mathcal{G}$ in the direction X is given by:

$$\mathcal{G}'(E) \cdot X = \int_{\partial E} \left(2 \int_E K(x-y) \, dy + H(x) \right) X(x) \cdot \nu(x) \, d\mathcal{H}^{m-1},$$

where $H := \operatorname{div}_\tau(\nu)$ is the mean curvature of E and ν denotes the outward unit normal to ∂E .

2.5 Euler Lagrange Condition

By the method of Lagrange multipliers, optimizing the shape functional $\mathcal{G}(E)$ under volume constraint $|E| = v$ is equivalent to finding the unconstrained critical points of the shape functional

$$\mathcal{L}(E) := \mathcal{G}(E) - \lambda(|E| - v),$$

where $\lambda \in \mathbb{R}$. The first variation of the shape functional $\mathcal{L}(E)$ in the direction of a smooth vector field $X \in C_c^\infty(\mathbb{R}^m; \mathbb{R}^m)$ is given by:

$$d\mathcal{L}(E)[X] = \int_{\partial E} \left(2 \int_E K(x-y) \, dy + H(x) \right) X(x) \cdot \nu(x) \, d\mathcal{H}^{m-1} - \lambda \int_{\partial E} X(x) \cdot \nu(x) \, d\mathcal{H}^{m-1}$$

Then, a set E is a critical shape of the shape functional $\mathcal{G}(E)$ under volume constraint if $d\mathcal{L}(E)[X] = 0$ for every X preserving the volume, then this implies

$$2 \int_E K(x-y) dy + H(x) = \lambda \quad \forall x \in \partial E$$

Proposition 4. The ball B_{R_v} is a critical point under the fixed-volume constraint of the shape functional $\mathcal{G}$.

Proof. Let $V_{B_{R_v}}(x) := \int_{B_{R_v}} K(x-y) dy$. Now, we show that the function $V_{B_{R_v}}$ is constant on ∂B_{R_v} . Let $x_1, x_2 \in \partial B_{R_v}$, then there exists a rotation $R \in SO(m)$ such that $x_2 = Rx_1$. We have

$$V_{B_{R_v}}(x_2) = \int_{B_{R_v}} K(x_2 - y) dy = \int_{B_{R_v}} K(Rx_1 - y) dy$$

We put $y = Rz$, then

$$V_{B_{R_v}}(x_2) = \int_{B_{R_v}} K(Rx_1 - Rz) dz = \int_{B_{R_v}} K(R(x_1 - z)) dz$$

Since the kernel K is radial, then there exists a function $f: [0, +\infty[\rightarrow \mathbb{R}$ such that $K(R(x_1 - z)) = f(|R(x_1 - z)|)$. As the rotations preserve the norm, then $K(R(x_1 - z)) = f(|x_1 - z|) = K(x_1 - z)$, so $V_{B_{R_v}}(x_2) = V_{B_{R_v}}(x_1)$.

Consequently, $V_{B_{R_v}}$ is constant on ∂B_{R_v} . Since $H_{B_{R_v}}(x) = \frac{m-1}{R}$. Therefore, $V_{B_{R_v}} + H_{B_{R_v}}$ is constant on ∂B_{R_v} . As a result, $\mathcal{G}'(B_{R_v}) \cdot X = C \int_{\partial B_{R_v}} X(x) \cdot \nu(x) d\mathcal{H}^{m-1}$. The fact that X is volume-preserving, then $\int_{\partial B_{R_v}} X(x) \cdot \nu(x) d\mathcal{H}^{m-1} = 0$. Hence, $\mathcal{G}'(B_{R_v}) \cdot X = 0$. Therefore, the ball B_{R_v} is a critical point under the fixed-volume constraint of the shape functional $\mathcal{G}(E)$. $\square$

Problem. Since under the volume constraint, balls represent critical points of the shape functional $\mathcal{G}$. However, the fact that a ball is a critical point does not, in itself, determine whether it is a local minimum, a local maximum, or a saddle point. Furthermore, since the perimeter is not bounded above among sets of fixed volume, the shape functional $\mathcal{G}$ does not admit a global maximum at a ball. Thus, a ball can be a local minimum or rather a saddle point of the shape functional $\mathcal{G}$, depending, in particular, on the kernel considered and the value of the volume.

This remark motivates the numerical study of the shape functional $\mathcal{G}$ on four families of regular shapes of the planar having the same area.

3 Numerical Computation of the Shape Functional $\mathcal{G}(E)$ for Some Tested Regular Planar Shapes of the Same Area and $K(x) = \exp(-\|x\|^a)$

In this section, we seek to study the approximative value of the energy $\mathcal{G}(E)$ on the tested families of regular planar shapes of the same area.

Let X and Y be independent random variables uniformly distributed over a set $E \subset \mathbb{R}^m$ with area $|E|$. Their probability density functions are:

$$f_X(x) = \frac{1}{|E|} \chi_E(x) \quad \text{and} \quad f_Y(y) = \frac{1}{|E|} \chi_E(y).$$

The fact that X and Y are independent, then their joint density function is:

$$f_{X,Y}(x, y) = f_X(x)f_Y(y) = \frac{1}{|E|^2} \chi_E(x)\chi_E(y)$$

Since $\mathbb{E}[K(X-Y)] = \int_E \int_E K(x-y) f_{X,Y}(x, y) dx dy = \frac{1}{|E|^2} \int_E \int_E K(x-y) dx dy$, then $\int_E \int_E K(x-y) dx dy = |E|^2 \mathbb{E}[K(X-Y)]$. Using the Law of Large Numbers, $\mathbb{E}[K(X-Y)]$ can be estimated empirically by sampling N independent, identically distributed pairs (X_i, Y_i) uniformly from E :

$$\mathbb{E}[K(X - Y)] \approx \frac{1}{N} \sum_{i=1}^N K(X_i - Y_i).$$

Consequently, $\int_E \int_E K(x - y) dx dy \approx \frac{|E|^2}{N} \sum_{i=1}^N K(X_i - Y_i)$, which is called the Monte Carlo estimator.

So, for every shape E in the planar of fixed area s , we have

$$\mathcal{P}_K^\alpha(E) \approx \frac{s^2}{N} \sum_{i=1}^N \exp\left(-((X_{i1} - Y_{i1})^2 + (X_{i2} - Y_{i2})^2)^{\frac{\alpha}{2}}\right),$$

where $X_i = (X_{i1}, X_{i2})$ and $Y_i = (Y_{i1}, Y_{i2})$, $i = 1, \dots, N$ independent, identically distributed pairs uniformly from E . We develop MATLAB algorithms that compute the approximate value of $\mathcal{P}_K^\alpha(E)$ for the disk, the equilateral triangle, the square and regular hexagon, all having the same prescribed area and for $N = 100$. The following tables present the approximate value of the shape functional $\mathcal{G}(E)$ for different prescribed areas s .

Case $\alpha = 2$.

Table 1. Numerical values of $\mathcal{G}(E)$, for $s = 20$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$2\sqrt{(20\pi)}$	69.5799	85.4332
Hexagon	$2\sqrt{(40\sqrt{3})}$	57.4472	74.0944
Square	$4\sqrt{(20)}$	52.0420	69.9305
Eq. triangle	$6\sqrt{(20\sqrt{3})/\sqrt{3}}$	46.7512	67.1395

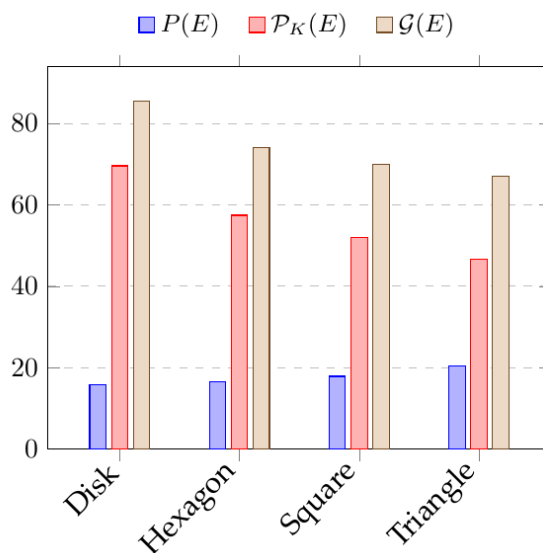


Fig. 1. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 1 for the disk, hexagon, square, and equilateral triangle ($s = 20$, $\alpha = 2$).

Table 2. Numerical values of $\mathcal{G}(E)$, for $s = 15$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$2\sqrt{(15\pi)}$	41.6533	55.3827
Hexagon	$2\sqrt{(30\sqrt{3})}$	40.6163	55.0332
Square	$4\sqrt{(15)}$	33.9180	49.4099
Eq. triangle	$6\sqrt{(15\sqrt{3})/\sqrt{3}}$	30.8548	48.5127

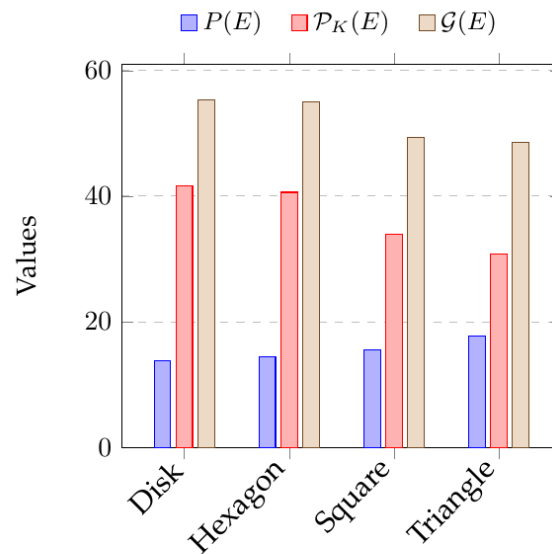


Fig. 2. Bar chart of the numerical values of $P(E)$, $P_K(E)$, and $G(E)$ of Table 2 for the disk, hexagon, square, and equilateral triangle ($s = 15$, $\alpha = 2$).

Table 3. Numerical values of $G(E)$, for $s = 3.5$ ($\alpha=2$).

Shape	$P(E)$	$P_K(E)$	$G(E)$
Disk	$2\sqrt{(3.5\pi)}$	6.0345	12.6664
Hexagon	$2\sqrt{(7\sqrt{(3)})}$	5.4870	12.4510
Square	$4\sqrt{(3.5)}$	4.8762	12.3595
Eq. triangle	$6\sqrt{(3.5\sqrt{(3)})/\sqrt{(3)}}$	4.3453	12.8745

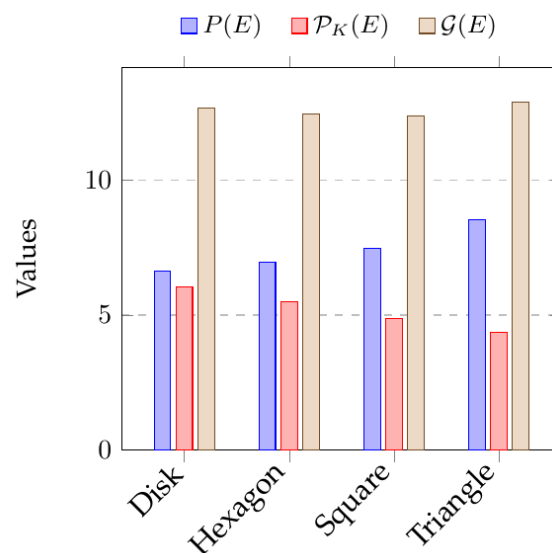


Fig. 3. Bar chart of the numerical values of $P(E)$, $P_K(E)$, and $G(E)$ of Table 3 for the disk, hexagon, square, and equilateral triangle ($s = 3.5$, $\alpha = 2$).

Table 4. Numerical values of $G(E)$, for $s = 3.2$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$2\sqrt{(3.2\pi)}$	5.1427	11.4840
Hexagon	$2\sqrt{(6.4\sqrt{(3)})}$	4.5935	11.2524
Square	$4\sqrt{(3.2)}$	4.5314	11.6868
Eq. triangle	$6\sqrt{(3.2\sqrt{(3)})/\sqrt{(3)}}$	4.2822	12.4376

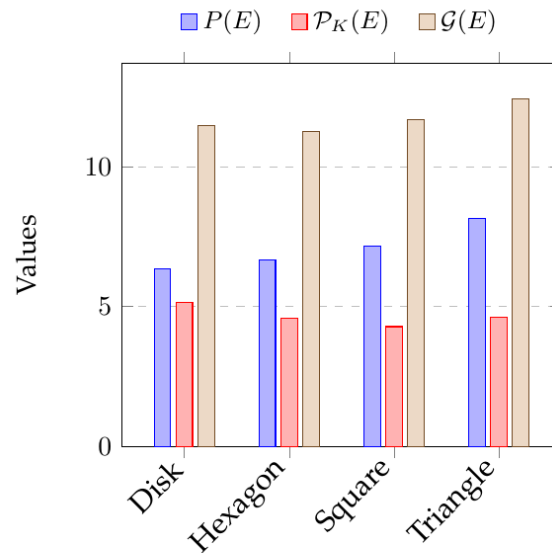


Fig. 4. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 4 for the disk, hexagon, square, and equilateral triangle ($s = 3.2$, $\alpha = 2$).

Table 5. Numerical values of $\mathcal{G}(E)$, for $s = \pi$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	2π	4.7584	11.0416
Hexagon	$\sqrt{(8\pi\sqrt{(3)})}$	4.6302	11.2280
Square	$4\sqrt{(\pi)}$	4.5065	11.5963
Eq. triangle	$6\sqrt{(\pi\sqrt{(3)})/\sqrt{(3)}}$	4.1333	12.2139

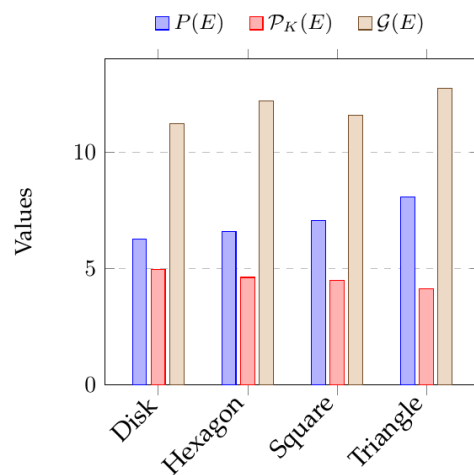
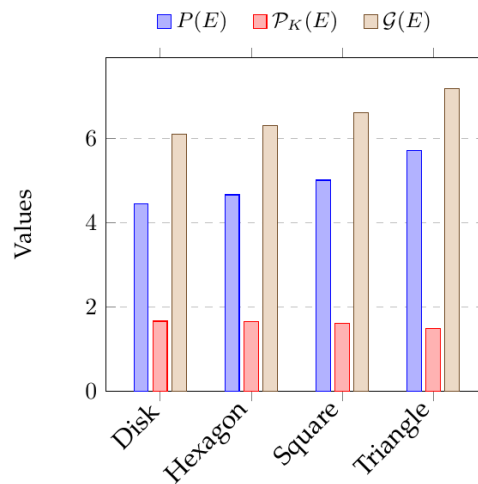


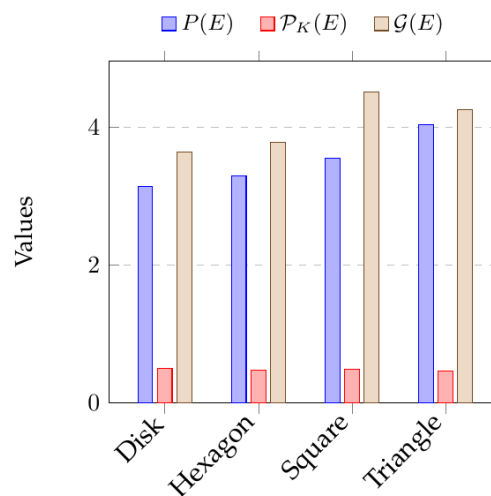
Fig. 5. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 5 for the disk, hexagon, square, and equilateral triangle ($s = \pi$, $\alpha = 2$).

Table 6. Numerical values of $\mathcal{G}(E)$, for $s = \pi/2$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$\sqrt{(2\pi)}$	1.66	6.1029
Hexagon	$2\sqrt{(\pi\sqrt{(3)})}$	1.6531	6.3185
Square	$4\sqrt{(\pi/2)}$	1.6005	6.6138
Eq. triangle	$3\sqrt{(2\pi)}/\sqrt{(\sqrt{(3)})}$	1.4850	7.1984

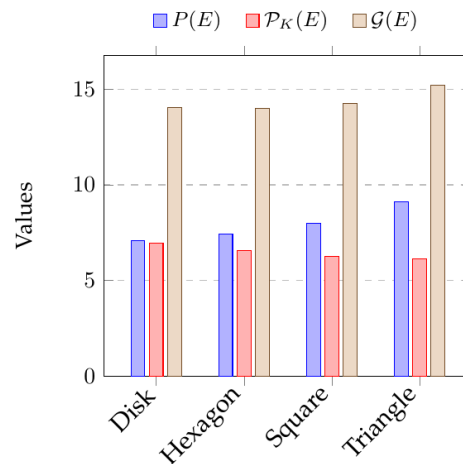
Fig. 6. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 6 for the disk, hexagon, square, and equilateral triangle ($s = \pi/2$, $\alpha = 2$).Table 7. Numerical values of $\mathcal{G}(E)$, for $s = \pi/4$ ($\alpha=2$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	π	0.4971	3.6387
Hexagon	$2\sqrt{(\pi/2\sqrt{(3)})}$	0.4800	3.7789
Square	$2\sqrt{(\pi)}$	0.4604	4.0053
Eq. triangle	$3\sqrt{(\pi)}/\sqrt{(\sqrt{(3)})}$	0.4683	4.25

Fig. 7. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 7 for the disk, hexagon, square, and equilateral triangle ($s = \pi/4$, $\alpha = 2$).

Case $\alpha = 1$.Table 8. Numerical values of $\mathcal{G}(E)$, for $s = 4$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$4\sqrt{\pi}$	6.9409	14.0307
Hexagon	$4\sqrt{2\sqrt{3}}$	6.5478	13.9926
Square	8	6.2538	14.2538
Eq. triangle	$12\sqrt{3}/\sqrt{3}$	6.1149	15.2310

Fig. 8. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 8 for the disk, hexagon, square, and equilateral triangle ($s = 4$, $\alpha = 1$).Table 9. Numerical values of $\mathcal{G}(E)$, for $s = 3.5$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$2\sqrt{3.5\pi}$	5.5354	12.1673
Hexagon	$2\sqrt{7\sqrt{3}}$	5.4587	12.4227
Square	$4\sqrt{3.5}$	5.0556	12.5389
Eq. triangle	$6\sqrt{3.5\sqrt{3}}/\sqrt{3}$	4.4379	12.9676

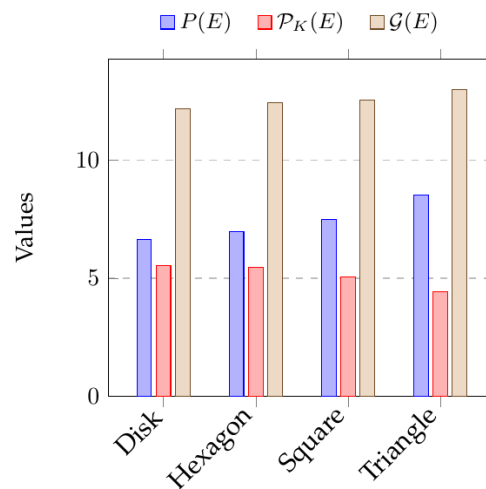
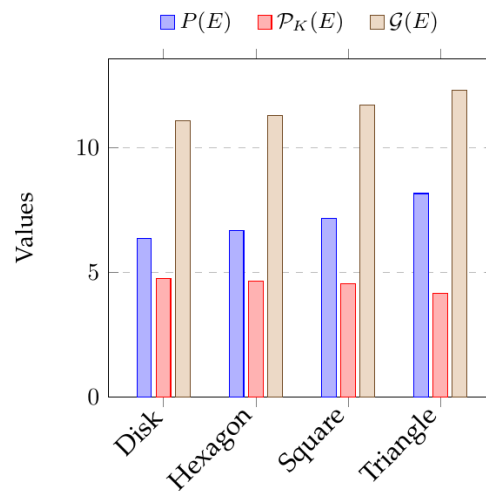
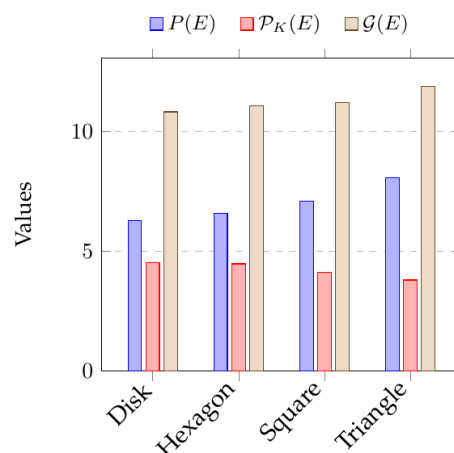
Fig. 9. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 9 for the disk, hexagon, square, and equilateral triangle ($s = 3.5$, $\alpha = 1$).

Table 10. Numerical values of $\mathcal{G}(E)$, for $s = 3.2$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$2\sqrt{(3.2\pi)}$	4.7304	11.0717
Hexagon	$2\sqrt{(6.4\sqrt{(3)})}$	4.6374	11.2963
Square	$4\sqrt{(3.2)}$	4.5409	11.6963
Eq. triangle	$6\sqrt{(3.2\sqrt{(3)})/\sqrt{(3)}}$	4.1565	12.3119

Fig. 10. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 10 for the disk, hexagon, square, and equilateral triangle ($s = 3.2$, $\alpha = 1$).Table 11. Numerical values of $\mathcal{G}(E)$, for $s = \pi$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	2π	4.5449	10.8281
Hexagon	$\sqrt{(8\pi\sqrt{(3)})}$	4.4833	11.0811
Square	$4\sqrt{(\pi)}$	4.1210	11.2108
Eq. triangle	$6\sqrt{(\pi\sqrt{(3)})/\sqrt{(3)}}$	3.8054	11.8860

Fig. 11. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 11 for the disk, hexagon, square, and equilateral triangle ($s = \pi$, $\alpha = 1$).Table 12. Numerical values of $\mathcal{G}(E)$, for $s = \pi/2$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	$\sqrt{2\pi}$	1.5743	6.0172
Hexagon	$2\sqrt{\pi\sqrt{3}}$	1.4715	6.1369
Square	$4\sqrt{\pi/2}$	1.3239	6.3372
Eq. triangle	$3\sqrt{2\pi}/\sqrt{\sqrt{3}}$	1.2913	7.0051

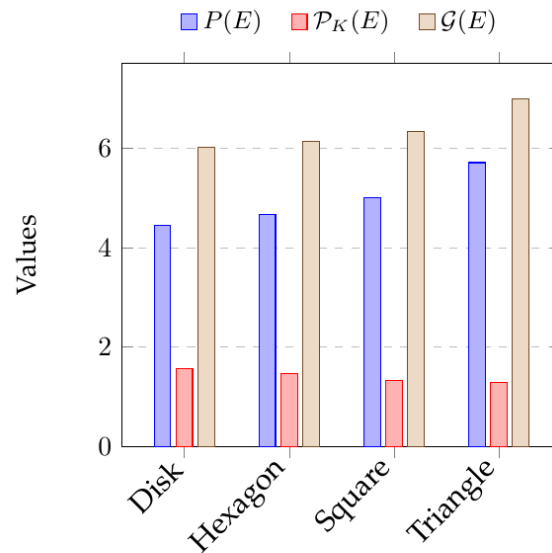


Fig. 12. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 12 for the disk, hexagon, square, and equilateral triangle ($s = \pi/2$, $\alpha = 1$).

Table 13. Numerical values of $\mathcal{G}(E)$, for $s = \pi/4$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	π	0.5028	3.6444
Hexagon	$2\sqrt{\pi/2\sqrt{3}}$	0.4071	3.7060
Square	$2\sqrt{\pi}$	0.3889	3.9338
Eq. triangle	$3\sqrt{\pi}/\sqrt{\sqrt{3}}$	0.3800	4.4196

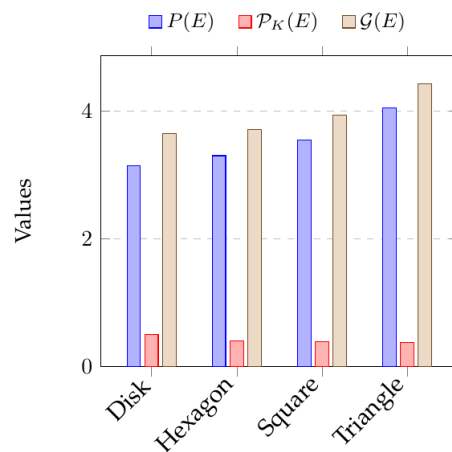
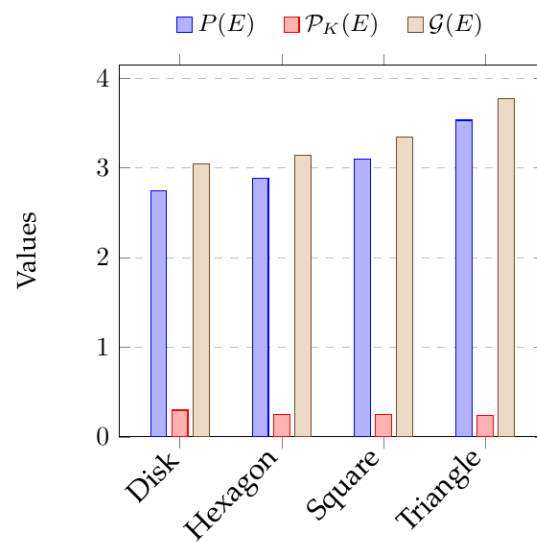


Fig. 13. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 13 for the disk, hexagon, square, and equilateral triangle ($s = \pi/4$, $\alpha = 1$).

Table 14. Numerical values of $\mathcal{G}(E)$, for $s = 0.6$ ($\alpha=1$).

Shape	$P(E)$	$\mathcal{P}_k(E)$	$\mathcal{G}(E)$
Disk	2.7459	0.2988	3.0447
Hexagon	2.8834	0.2528	3.1362
Square	3.0984	0.2487	3.3471
Eq. triangle	3.5314	0.2366	3.7680

Fig. 14. Bar chart of the numerical values of $P(E)$, $\mathcal{P}_k(E)$, and $\mathcal{G}(E)$ of Table 14 for the disk, hexagon, square, and equilateral triangle ($s = 0.6$, $\alpha = 1$).

Remark 1. Based on the numerical experiments above, restricted to the four tested families of regular shapes, we conclude that the results suggest the existence of a threshold area s_0 that depends on the choice of α : $s_0 \in [\pi, 3.2[$ for $\alpha = 2$, while $s_0 \in [3.5, 4[$ for $\alpha = 1$. Below this threshold area s_0 , the disk minimizes the shape functional $\mathcal{G}(E)$ among the four tested families of regular shapes.

3.1 Analysis of the energy difference $\Delta G_s(E) = G_s(E) - G_s(\text{Disk})$ for $\alpha = 2$

In this subsection, we plot the energy difference $\Delta G_s(E) = G_s(E) - G_s(\text{Disk})$ as a function of the area parameter s , for the regular hexagon, square, and equilateral triangle and for $\alpha = 2$.

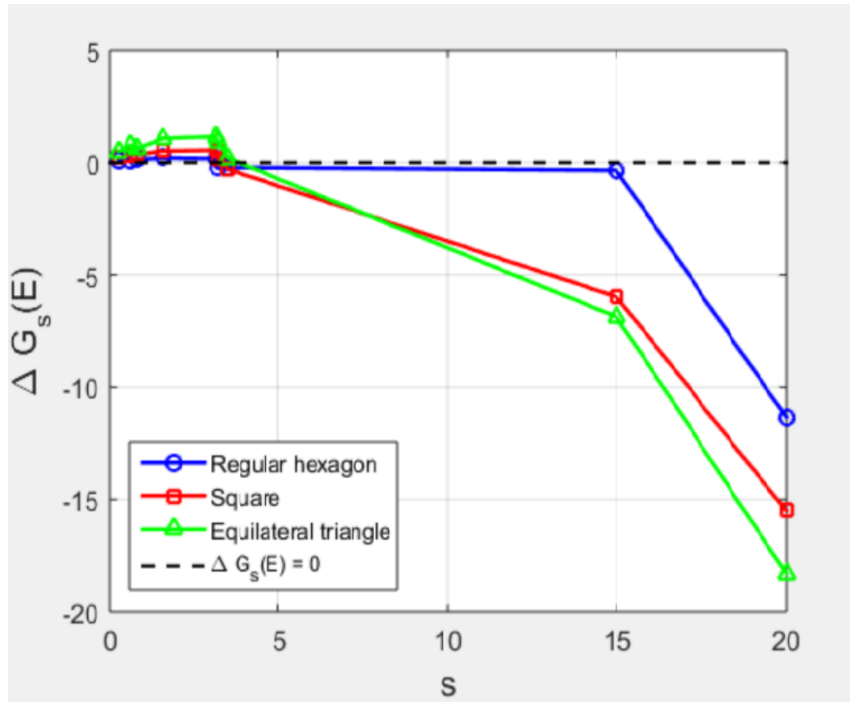


Fig. 15. Energy difference $\Delta G_s(E) = G_s(E) - G_s(\text{Disk})$ as a function of the area parameter s , for the regular hexagon, square, and equilateral triangle ($\alpha = 2$).

4 Study the Minimization of the Augmented Shape Functional of the Shape Functional G

In this section, we look for studying the minimization of the augmented shape functional denoted by $\mathcal{G}_R(E)$ on the admissible shape $\mathcal{A}_v$ and defined as follows:

$$\mathcal{G}_R(E) := \mathcal{G}(E) + 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E), \quad (3)$$

where $\alpha(E) := \inf_{y \in \mathbb{R}^m} \frac{|E \Delta (B_{R_v} + y)|}{|E|}$ the Fraenkel asymmetry of the shape E .

Proposition 5. For any admissible shapes E , we have the following:

$$\mathcal{P}_K(B_{R_v}) \leq \mathcal{P}_K(E) + 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E), \quad (4)$$

where $R_v = \left(\frac{v}{|B|}\right)^{\frac{1}{m}}$.

Proof. Remember that from Proposition 1 and the fact that the ball B_{R_v} maximizes the shape functional $\mathcal{P}_K$ on the admissible shapes $\mathcal{A}_v$, it gives that:

$$(\forall E \in \mathcal{A}_v)(\forall y \in \mathbb{R}^m) \quad \mathcal{P}_K(B_{R_v}(y)) \leq \mathcal{P}_K(E) + 2 \|K\|_{L^1(\mathbb{R}^m)} |E \Delta B_{R_v}(y)|.$$

The fact that the functional $\mathcal{P}_K$ is invariant up to translation, it yields

$$(\forall E \in \mathcal{A}_v)(\forall y \in \mathbb{R}^m) \quad \mathcal{P}_K(B_{R_v}) \leq \mathcal{P}_K(E) + 2 \|K\|_{L^1(\mathbb{R}^m)} |E \Delta B_{R_v}(y)|.$$

Taking the infimum over all $y \in \mathbb{R}^m$, we obtain $(\forall E \in \mathcal{A}_v) \mathcal{P}_K(B_{R_v}) \leq \mathcal{P}_K(E) + 2 \|K\|_{L^1} \inf_{y \in \mathbb{R}^m} |E \Delta B_{R_v}(y)|$. Consequently, $(\forall E \in \mathcal{A}_v) \mathcal{P}_K(B_{R_v}) \leq \mathcal{P}_K(E) + 2v \|K\|_{L^1} \alpha(E)$. ▀

The following proposition gives an upper bound of the Fraenkel asymmetry of the minimizer E^* of the shape functional $\mathcal{G}(E)$.

Proposition 6. *There exists a constant universal $C(m) > 0$ such that the Fraenkel asymmetry of the minimizer E^* satisfies the following:*

$$\alpha(E^*) \leq \left(\frac{v \|K\|_{L^1(\mathbb{R}^m)}}{C(m)P(B_{R_v})} \right)^{\frac{1}{2}}.$$

Proof. Thanks to the quantitative isoperimetric inequality ([2, 7]), then there exists a constant dimensional $C(m) > 0$ such that for every shape E in the admissible shapes $\mathcal{A}_v$, we have $P(E) \geq P(B_{R_v}) + C(m)P(B_{R_v})\alpha(E)^2$. In particular for E^* the minimizer of the shape functional $\mathcal{G}$, we get $P(E^*) \geq P(B_{R_v}) + C(m)P(B_{R_v})\alpha(E^*)^2$.

Since E^* is a minimizer of the shape functional $\mathcal{G}(E)$ on $\mathcal{A}_v$, then $\mathcal{G}(B_{R_v}) \geq \mathcal{G}(E^*) \geq P(E^*)$. Therefore,

$$\mathcal{P}_K(B_{R_v}) \geq C(m)P(B_{R_v})\alpha(E^*)^2 \quad (5)$$

On the other hand, we have

$$\begin{aligned} \mathcal{P}_K(B_{R_v}) &= \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{B_{R_v}}(x) \chi_{B_{R_v}}(y) K(x-y) dx dy = \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \chi_{B_{R_v}-y}(x) \chi_{B_{R_v}}(y) K(x) dx dy \\ &= \int_{\mathbb{R}^m} \chi_{B_{R_v}}(y) \left(\int_{\mathbb{R}^m} \chi_{B_{R_v}-y}(x) K(x) dx \right) dy \quad (\text{apply Fubini's theorem}) \\ &\leq \int_{\mathbb{R}^m} \chi_{B_{R_v}}(y) \left(\int_{\mathbb{R}^m} K(x) dx \right) dy \leq v \|K\|_{L^1(\mathbb{R}^m)}. \end{aligned} \quad (6)$$

Combining (5) and (6), we obtain $C(m)P(B_{R_v})\alpha(E^*)^2 \leq v \|K\|_{L^1(\mathbb{R}^m)}$. Consequently,

$$\alpha(E^*) \leq \left(\frac{v \|K\|_{L^1(\mathbb{R}^m)}}{C(m)P(B_{R_v})} \right)^{\frac{1}{2}}. \quad \square$$

Theorem 2. *For any set E in the admissible shapes $\mathcal{A}_v$, we have the following:*

$$\mathcal{G}_R(B_{R_v}) \leq \mathcal{G}_R(E).$$

Proof. Remember that from Proposition 5, we have $\mathcal{P}_K(B_{R_v}) \leq \mathcal{P}_K(E) + 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E)$ for every $E \in \mathcal{A}_v$. The fact that the perimeter is minimized by balls, this implies that:

$$(\forall E \in \mathcal{A}_v) \quad \mathcal{P}_K(B_{R_v}) + P(B_{R_v}) \leq \mathcal{P}_K(E) + P(E) + 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E).$$

Then, $\mathcal{G}(B_{R_v}) \leq \mathcal{G}(E) + 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E)$. Since $\alpha(B_{R_v}) = 0$, then for every $E \in \mathcal{A}_v$

$$\mathcal{G}_R(B_{R_v}) \leq \mathcal{G}_R(E).$$

Thus, the ball B_{R_v} is a global minimizer of the augmented shape functional $\mathcal{G}_R$ on the admissible shapes $\mathcal{A}_v$. $\square$

Corollary 3. *The gap $\mathcal{G}(B_{R_v}) - \mathcal{G}(E^*)$ satisfies the following:*

$$0 \leq \mathcal{G}(B_{R_v}) - \mathcal{G}(E^*) \leq 2 \left(\frac{v^3 \|K\|_{L^1(\mathbb{R}^m)}^3}{C(m)P(B_{R_v})} \right)^{\frac{1}{2}},$$

where E^* is a minimizer of the shape functional $\mathcal{G}$ on $\mathcal{A}_v$.

Proof. Since E^* is the minimizer of the shape functional $\mathcal{G}(E)$, then $\mathcal{G}(E^*) \leq \mathcal{G}(B_{R_v})$. Thanks to the definition of Fraenkel asymmetry and Theorem 2, we obtain $\mathcal{G}(B_{R_v}) \leq \mathcal{G}_R(E^*)$. Hence, $\mathcal{G}(E^*) \leq \mathcal{G}(B_{R_v}) \leq \mathcal{G}_R(E^*)$. So,

$$0 \leq \mathcal{G}(B_{R_v}) - \mathcal{G}(E^*) \leq 2v \|K\|_{L^1(\mathbb{R}^m)} \alpha(E^*).$$

From Proposition 6 we can conclude:

$$0 \leq \mathcal{G}(B_{R_v}) - \mathcal{G}(E^*) \leq 2 \left(\frac{v^3 \|K\|_{L^1(\mathbb{R}^m)}^3}{C(m)P(B_{R_v})} \right)^{\frac{1}{2}}. \quad \square$$

5 Conclusion

In this work, we investigated a minimization problem for a shape functional defined as the sum of the perimeter in the sense of Caccioppoli–De Giorgi and a non-local interaction term generated by a regular kernel. The study was conducted on a class of measurable sets with a finite perimeter and a fixed volume.

In the first part of this paper, we established the existence of a minimizer for the functional under consideration. This result relies primarily on the lower semi-continuity of the functional and on the compactness theorem within the space of functions of bounded variation (BV). This approach, thus, provides a variational framework that guarantees the existence of an optimal shape within the admissible class.

Next, we investigated the first shape derivative of the shape functional under a fixed-volume constraint. We then demonstrate that when the interaction kernel is radial, the ball satisfies this optimality condition. Consequently, the ball is a critical point of the shape functional with respect to all admissible volume-preserving perturbations.

The theoretical analysis was complemented by a two-dimensional numerical study based on the Monte Carlo method. Four families of regular shapes with equal area were considered: the disk, the regular hexagon, the square, and the equilateral triangle. For the values tested, the results suggest the existence of a threshold area s_0 within the interval $[\pi, 3.2[$ for $\alpha = 2$ and within the interval $[3.5, 4[$ for $\alpha = 1$. For sampled values below this threshold, the disk is the minimizer of the shape functional among the four families tested.

Also, we controlled the Fraenkel asymmetry of the minimizer E^* by a term which depends on the volume v and $\|K\|_{L^1(\mathbb{R}^m)}$. An augmented shape functional $\mathcal{G}(E)$ was then considered by adding a penalization term, and we proved that the ball of volume v is a minimizer of the augmented shape functional.

Several avenues exist for extending this work. One initial direction involves theoretically establishing the existence of a critical threshold separating different minimization regimes. It would also be interesting to extend the study to other classes of interaction kernels. Finally, from a numerical standpoint, another study will subsume the use of deterministic quadrature or cubature methods, complementing the Monte Carlo method.

DECLARATIONS

Funding. This research received no external funding.

Conflicts of interest. The authors declare that they have no competing interests.

Ethics approval. Not applicable. This study did not involve human participants or animals.

Informed consent. Not applicable. This study did not involve human participants.

Data and code availability. The MATLAB code used to perform the numerical experiments in this study is available from the corresponding author upon reasonable request.

Author contributions. The author conceived the study, developed the mathematical analysis, performed the numerical experiments in MATLAB, and wrote the manuscript.

Use of artificial intelligence. Author declare that no AI tools were used beyond language editing.

Acknowledgements. The author would like to express his sincere gratitude to his thesis supervisor for valuable guidance, continuous support, and constructive suggestions throughout this research. The author also wishes to thank the anonymous reviewer for the careful reading of the manuscript and for the valuable comments and suggestions that helped improve the quality of the paper.

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